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AL-BAYRÛNÎ'S LETTER ON ABÛ NAŞR MANŞÛR'S DEMONSTRATION OF THE SINE LAW

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The little tract of Al-Bayrûnî, presented here in its Arabic text as well as Turkish and English translations, is, as is attested by its closing sentence, a letter written by him to someone called Abû Sa'îd. It occupies pp. 134b-136a in a volume made up of a collection of Arabic manuscripts. The volume is in Leiden, in *Bibliotheca Academia Lugduno Batavae* and is registered there under number 1007 (Codex 168 Golii). This short article of Al-Bayrûnî was brought to the attention of the scholarly world through its German translation by Suter.¹

As its opening words indicate, this letter of Al-Bayrûnî announces and brings to light certain contributions of Abû Naşr Manşûr ibn 'Alî ibn 'Irâq to plane and spherical trigonometry. This contribution is Abû Naşr's proof for the sine law, and it contains the theorems covering all the special cases of the sine law. Abû Naşr Manşûr was the teacher and patron of Al-Bayrûnî. He was a prince of the family of the dynasty which ruled Western Khwârazm until 995 A.D., and he was an outstanding mathematician and astronomer.

It is not stated in this article explicitly that the law of sines was discovered by Abû Naşr. For the very first word of the tract relevant to this question is "derivation" and not "discovery". It would seem therefore possible from this wording that the law itself may have been of earlier date and that Abû Naşr may simply have supplied new proofs for it. Or, again, the connotation of this pithy statement may be that Abû Naşr's contribution was somewhat more substantial, though he was not fully responsible for the announcement of the law for the first time.

¹ Heinrich Suter, "Zur Trigonometrie der Araber", *Bibliotheca Mathematica*, series 3, volume 10, Heft 2, Leipzig 1910, pp. 156-160.

Al-Bayrûnî has another work which is still extant, though considered lost until recently,² in which he takes up this and other closely related topics in greater detail, as we gather from Naşîruddîn at-Tûsî's reference to it. This book is entitled *Maqâlîdu 'Ilmî Hay'âtî mâ Yaḥdithu fî Basīṭi'l Kuratî wa Ghayrihi* (*The Keys to the Knowledge of the Properties of Superficial Spherical and Other Figures*). According to Naşîruddîn, in fact, Al-Bayrûnî states in this work that it was, with the greatest likelihood, Abû Naşr who was first able to establish this law in its most general form, covering all its special cases, though Abû'l Wafâ Muḥammad ibn Muḥammad al Buzjânî and Abû Maḥmûd Ḥâmid ibn al Khidr al Khujandî also claimed priority in this respect.³

It is true that Naşîruddîn seems somewhat non-committal in this question.⁴ But Al-Bayrûnî's opinion in this matter should certainly carry considerable weight, and it is therefore quite likely that Abû Naşr generalized and extended the law of sines by discovering some of its special cases for the first time, possibly those dealing with plane triangles.

This may possibly have been the case. For apparently Naşîruddîn at-Tûsî was interested in the history of the subject especially in so far as spherical trigonometry is concerned. His text in fact contains considerable detail on the proofs found by various mathematicians for the sine law as applied to spherical triangles.⁵ But he has nothing to say about contributions of mathematicians to the subject of the sine law for plane triangles.

It seems reasonable, indeed, as we shall see a bit further on, to conjecture that the sine law was generalized and extended⁶ to plane triangles after having first been established for the case of spherical triangles. We obviously cannot be sure, in the face of lack of unequivocal and conclusive evidence in our sources, however, that the honor of making such a generalization and extension belonged solely to Abû Naşr Maṣṣûr.

² See, *Isis*, vol. 64, No. 225, 1973, p. 69 (item 1007).

³ Naşîruddîn at-Tûsî, *Kitābu Shakli'l-Qaṭ'ā'*, ed. and tr. A. Caratheodori, Istanbul 1891, text, p. 108, tr., p. 140.

⁴ Naşîruddîn, *op. cit.*, text, pp. 108, 125, tr., pp. 140, 162-163.

⁵ Naşîruddîn, *op. cit.*, text, pp. 108-125, tr., pp. 140-163.

The present little article, or letter, is not included in Al-Bayrūnī's detailed list of his own works, although, if it is true, as asserted by Suter, that the letter was written to Abū Sa'īd as-Sijzī (or Sijazī) (d. ca. 1024), it should be of earlier date than the said list, which was prepared in 1036. In the manuscript itself only the name Abū Sa'īd occurs. I do not know what evidence Suter has for considering this to refer to Abū Sa'īd as-Sijzī.

Apart from these considerations, however, it is clear from the first sentence of the text that Al-Bayrūnī wrote it before he had reached the age of 23, i.e., before Abū 'Abdullāh Muḥammad ibn Aḥmad ibn 'Irāq, the last king of the family of Ibn 'Irāq, rulers of Western Khwārazm, was overthrown by the ruler of Eastern Khwārazm in 995. For Al-Bayrūnī refers to him here, in his opening sentence, as a man still living and as a ruler still in power.

It is true that Suter translates the key expression here, *mawlā Amīri'l-mu'minīn*, as the "freed slave of the *amīr* of the believers", and this would render the chronological connotation of that statement vague. Al-Bayrūnī uses this expression quite frequently when speaking of Abū Naṣr Maṣṣūr. Sachau also translated this word *mawlā*, used by Al-Bayrūnī when speaking of Abū Naṣr Maṣṣūr, as "freed slave", and he decided that the term *Amīru'l-mu'minīn* occurring together with it should refer here to the Samanid king of the time.⁶

Such a reference is quite unusual, however, and it is rather odd that Al-Bayrūnī should have repeatedly used it in the sense of "freed slave" when speaking of Abū Naṣr Maṣṣūr. The term *amīru'l-mu'minīn* was used frequently and for any ruler, and one of the meanings of the word *mawlā* is "close relative". It is known, on the other hand, that, as the part of his name "Ibn 'Irāq" also indicates, Abū Naṣr Maṣṣūr belonged to the ruling family of Western Khwārazm. There should be no doubt therefore that the meaning of this expression, as used in this context by Al-Bayrūnī, is 'the close relative of the king of Khwārazm', as I have interpreted it.

It is quite understandable of course that Al-Bayrūnī should have forgotten to include this letter of his youthful days in the list of his works prepared at a considerably later date or that he should not have

⁶ See, Al-Bayrūnī, *Al-Āthāru'l-Bāqīya*, ed. Sachau, Introduction, p. 33.

* even considered it necessary to include such a letter concerning someone else's work, learned though it may be, in the list of his own publications. The last sentence of this tract constitutes sufficient evidence that it was written by Al-Bayrûnî, and, as we have just seen, we are in possession of clear evidence that the subject of this letter was one which was of great interest to him and to which he referred in greater detail in at least one of his more substantial works. We should therefore not doubt the authenticity of its attribution to Al-Bayrûnî.

As mentioned above, Naşîruddîn speaks of the history of the sine law in detail only in connection with spherical triangles. Moreover, the figures concerning these theorems that occur in the text of Naşîruddîn's work which has survived were considered often wanting in clarity. These details were consequently omitted in Von Braunnühl's history of trigonometry in Islam, and this had resulted in the creation of a slightly distorted image of the course of progress of trigonometry in Islam. For it had brought the thirteenth century somewhat into prominence at the expense of underestimating certain achievements, especially in plane trigonometry, belonging to the earlier centuries.⁷ Al-Bayrûnî's present short tract was one of the most important documents serving to correct that impression.

As this letter of Al-Bayrûnî is thus obviously of considerable significance to the history of trigonometry in Islam, it was deemed advisable to publish it, although, as mentioned before, Suter has already made a full German translation of it available. I also thought it might not be superfluous to publish on this occasion an English translation of it too, as my reading and understanding of the text differ in certain respects from those of Suter.

The manuscript is badly worm-eaten in parts. Fortunately, however, the context makes it possible to emend and complete, without much difficulty, the small lacunas that have thus come into being. The words and letters added in order to fill these gaps have been placed in square brackets in the printed text. In four places the word *sine* is missing in the manuscript, and there is no question of their not having been left out by mistake. These have been added therefore to the text and inserted in braces. Likewise, toward the end of the

⁷ Suter, *op. cit.*, pp. 156-157.

manuscript something like a full line was skipped by the copyist. This is quite obvious from the context and also because the missing part occurs between two expressions in the form "the sine of the angle", and the manuscript has this expression only once and lacks the intervening words. The missing part of about one line which has been added has also been inserted within braces in the printed text. The manuscript is otherwise quite legible and clear, and it contains only two other errors which have been corrected as can be seen from the footnotes.

Suter makes the conjecture that the last demonstration, or, as I call it, the fifth special case, set forth in the article is a later addition, appended possibly "by Al-Bayrūnī himself".⁸ This is a somewhat strange statement, for the whole letter was penned by Al-Bayrūnī anyway. Suter apparently means also to say that this last proof is probably a contribution to the subject due to Al-Bayrūnī. But such a claim too would not seem much to the point. For, as Suter himself says, this proof is not different at all from the other proofs set forth in the text.

Suter's vaguely phrased conjecture is based mainly on the lettering of the triangle connected with this last special case as ADG (our figure 5), as compared to the lettering ABG that occurs in the triangles of the preceding parts of the text. He also says that this last paragraph is briefer and more concise as compared to the earlier treatments of the examples of triangles, and he adds incidentally that this part of the text is defective, for certain parts of ratio's involved in the intended proof are missing.

Suter's views concerning this last part of the text are certainly mistaken and unsatisfactory. I have come to the conclusion that what has misled Suter in the first place is his interpretation of the figure of the text which I have reproduced here as figure 4 and which belongs to the one-before-last case dealt with in the text. The lines AH and AG in this figure are quite close to each other in the figure of the manuscript, and the part of the figure where the letters H and G are located is worm-eaten.

⁸ Suter, *op. cit.*, p. 160, note 2.

Suter takes G to be on the left-hand side of H (figure 6 in Suter). This makes the triangle ABG an obtuse-angled triangle. But the final figure belonging to the "fifth special case" (the manuscript's and our figure 5, and Suter's figure 7) also represents an obtuse-angled triangle, and the two final cases dealt with thus become the same. That is why the final case appears to Suter to be some kind of a repetition of the one coming immediately before it.

This confusion is also caused partly by the lack of clarity of the same figure 4 as to whether the line to the left or whether the one to the right (i. e., AH and AG) is perpendicular to the base line. I have come to decide that G belongs to the point on the right and H to the point on the left, the point H being the foot of the perpendicular to the base line as explicitly stated in the text. This makes the one-before-last figure of the text (my figure 4) an acute-angled triangle and the last figure an obtuse-angled triangle as is clear from the figure of the manuscript anyway (figure 5 of our printed text). The two special cases to which these figures refer, i. e., the last two proofs in the text, thus cease to be repetitive, and the doubts raised by Suter become pointless.

This way of interpretation of these two figures for plane triangles finds its exact parallel in the two previous figures of the handwritten text (our figures 2 and 3) for spherical triangles, one of which represents an acute-angled spherical triangle and the other an obtuse-angled spherical triangle. As to Suter's observation that the last paragraph of the text is shorter and more concise as compared to the earlier sections dealing with plane triangles, this simply arises from two circumstances.

It is firstly due to the circumstance that a text of about one line was left out by the copyist in the last paragraph, and it should secondly be pointed out that the earlier sections dealing with plane triangles are a little lengthy because in these sections certain concepts relating to plane trigonometry are elucidated by giving certain definitions. But these definitions and clarifications apply to the last paragraph as well. It is natural therefore that these should not have been repeated, and this too makes the last paragraph shorter and more concise.

Moreover, the terminating sentence of the tract clearly seems to round up the whole article, and there is no trace of a phraseology

resembling a terminating statement coming before the fifth special case. It may also be pointed out in this connection that this tract seems to have been conceived as composed of two major parts, viz., the part dealing with spherical triangles, and the part dealing with plane triangles. For the expression Q. E. D. comes only at the ends of these major parts and not at the ends of the other more specific special cases. And as the second of these expressions comes at the end of the last paragraph, this too may be taken to indicate that the last paragraph was not intended as an appendage to an otherwise complete formal treatment. Thus the various parts of the letter fit together rather nicely and contribute towards the formation of a well-integrated whole not only from the standpoint of the subject matter but also from the viewpoint of phraseology and literary expression especially in what concerns the opening and closing sentences.

It is true that what I call the fifth case starts with the words *ibâretun ukhrâ* which Suter interprets to mean something like an alternative proof. But this could mean "another exposition" or "a complementary relation". Such an interpretation would seem reasonable. At any rate, these two cases need not necessarily have been conceived as two cases sharply distinguished from each other. Indeed, the corresponding two cases for spherical triangles are condensed into a single special case in the wording of the letter, although this part of the text is illustrated with two different figures.

As it shall presently be pointed out, one feature of this text is that its parts dealing with spherical triangles are more condensed than the parts treating plane triangles. And this feature holds true for the case concerning the right-angled plane triangle and that dealing with obtuse-angled plane triangle. Nevertheless, they are both represented by a single figure in the manuscript. Thus, in a sense, Al-Bayrûnî did not feel the need of drawing too clear a line of demarcation between these two cases either.

Under these circumstances, or under any circumstance, for that matter, it is quite natural that the proofs of the last two cases should be of exactly the same nature. This identity of method of proof would not constitute repetitiveness but would simply indicate that in a sense these two parts of the text are consistent or consonant with each other.

* For, as they are complementary, the same method of proof is simply applied in both cases.

As a matter of fact, I was able to supply the missing part of about a line in this final paragraph by using words and expressions taken from the special case preceding it, and these are found in more or less the same form in the part of the text dealing with spherical triangles. For the same feature occurs in the proofs given for those cases too. To be more exact, this is the common feature of the four special cases dealing with acute-angled and obtuse-angled plane and spherical triangles, i. e., of all the proofs with the exception of those concerning right-angled plane and spherical triangles.

I have changed, in the text, the letters referring to the geometrical figures to the corresponding letters of the Latin alphabet. This was not done only in order to simplify the printing but also to make one set of figures sufficient for both the text and its translations. The manuscript is one continuous text. I have divided it into paragraphs, and the punctuation is all mine. I have added dots of letters occasionally missing in the manuscript and have omitted some of the diacritical marks of the handwritten text. The manuscript has no *hamza* signs; the few which appear in the printed text do not belong to the manuscript.

The first figure of the manuscript does not conform well to the requirements of the conditions of perspective involved in the representation of the sector of sphere it stands for. The three dimensional situation represented is not clear, therefore, and it is not easy to visualize the lines occurring on it. Such ambiguity of figures is rather common in such manuscript texts due especially to careless reproduction by copyists not familiar with the subject. I have replaced this figure of the manuscript by its improved version supplied by Suter. As this improved figure represents what in reality the text needs and very likely is more similar to what it originally had, I have not reproduced the vague figure of the manuscript as Suter has done. For this reason, figure 1 of our printed text corresponds to figure 2 of Suter's translation.

As previously pointed out, figure 4 in the manuscript, which is figure 4 in our text also, is not very clear either and is worm-eaten in one part. This figure of ours is different from the corresponding figure

given by Suter (figure 6), as the points G and H are differently situated in them. Moreover, Suter has split this figure 4 of the manuscript into two in order to have an independent additional figure for the special case dealing with a right-angled plane triangle. I have not followed Suter in this respect and have remained faithful to the manuscript. For it is easy enough to follow the case of right-angled and acute-angled plane triangles from this same figure, and the phraseology of the text refers to a single figure for both special cases anyway. The result of all this is that our printed text has five figures, as the manuscript itself, whereas Suter's translation contains seven figures.

Figures 2, 3, and 4 of the present printed text contain in the manuscript certain straight-line segments placed by the sides of these figures. Such appendages to geometrical figures are found in other medieval manuscripts also. They apparently served the purpose of placing side by side and representing such magnitudes, at times non-homogeneous, as sines of angles and lengths of line segments or arcs as magnitudes of the same kind and had the function of expressing them in reference to a common scale. These line segments have been omitted in the figures of our text, as was also done by Suter.

It is interesting to note that in this tract the radii of the spheres or circles to which the sines are referred are taken to be unity throughout. This interesting convention, which apparently never became widespread, must therefore have dated back to the early years of Al-Bayrûnî.

Naşîruddîn at-Ṭûsî associates and attributes the adoption of the value of unity for the radius in trigonometry to Al-Bayrûnî.⁹ If this means, as Naşîruddîn seems to imply, that such a convention was first proposed and adopted by Al-Bayrûnî, such a datum would be of special interest to us from the vantage point of our present manuscript also. For this would mean that though the proofs incorporated in the present tract belonged to Abû Naşr Mañşûr, Al-Bayrûnî may have had his own way in the matter of their formal presentation. This too, incidentally, would run counter to what we may call Suter's

⁹ Naşîruddîn, *op. cit.*, text, p. 136, tr., p. 175. See also, text, pp. 140-142, tr., pp. 180-184, and, Salih Zeki, *Âthâr-i Bâqiye*, vol. 1, Istanbul 1329 H. (1913 A. D.), p. 61.

more or less tacit assumption that in this letter Al-Bayrûnî probably simply reproduces a text belonging to his master Abû Naşr, possibly adding to it the final paragraph.¹⁰

Naşîruddîn reproduces four proofs of Abû Naşr Manşûr for the theorems of the sine law for spherical triangles.¹¹ The texts of none of these is the same as the present manuscript text. This may again be interpreted to indicate that our manuscript text was not a copy of Abû Naşr Manşûr's own wording. We cannot be sure, however, that any of the texts given by Naşîruddîn was a faithful copy of the original either. In fact, one of these proofs is specifically stated by Naşîruddîn to belong to both Abû Naşr and Abû'l Wafâ "though their actual wordings are different".¹²

It may be noted that in these proofs too the radius is taken to be unity. But this is apparently because Naşîruddîn was not interested in quoting the texts of other mathematicians word for word but only in reproducing the specific nature of each proof. In fact the adoption of the value of the radius as unity is a common feature of all the different proofs connected with the sine law as reproduced by Naşîruddîn.

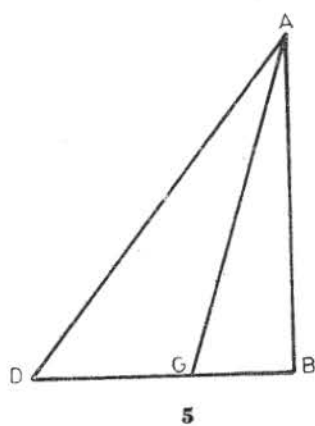
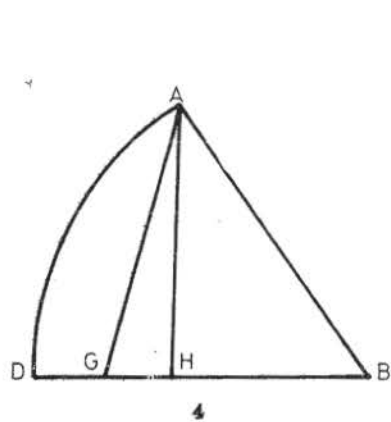
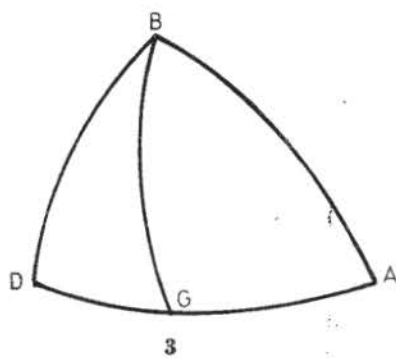
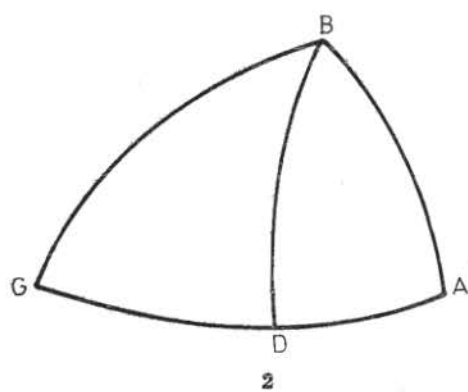
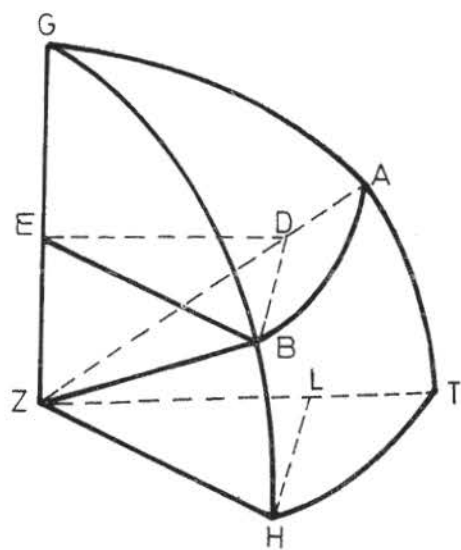
Another peculiarity of the present tract is that in it the law of sines for plane triangles is mentioned after the cases dealing with spherical triangles. Moreover, the equivalence of the measure of a plane angle and the arc of circle subtended by it is explicitly stated. In fact, this definition is set forth twice when treating the cases represented by figure 4, and likewise, the sine of a plane angle is defined on this occasion, whereas no such definitions are given for the case of a diedral angle, e.g., or rather, a very brief and partial reference is made thereto, although it may naturally be claimed that the latter case is less simple and more in need of clarification.

All this would seem to create the impression that the law of sines was first discovered for the case of spherical triangles and then extended and generalized to the case of plane triangles. Such a circumstance may in fact be said to be in conformity with the general

¹⁰ See above, p. 187 and note 8.

¹¹ Naşîruddîn, *op. cit.*, text, pp. 109-114, 121-122; tr. pp. 143-148, 160-162.

¹² Naşîruddîn, *op. cit.*, text, p. 111, tr., p. 145.



course of the history of trigonometry in Islam. For spherical trigonometry was much more in the foreground in the cultivation and progress of knowledge in this branch of mathematics, and this was due to the circumstance that work in trigonometry was motivated largely in its capacity of a mathematical tool needed by astronomy.

These definitions given in connection with plane angles in the text seem to indicate that a need was felt to refer a plane angle to the arc subtended by it and to associate trigonometric functions with such arcs rather than with the space bounded by two lines coming together in a point. The peculiar turn of such details in the present text would seem to suggest that plane trigonometry was, roughly speaking, in its stage of infancy at the time and that in utilizing trigonometric functions in plane triangles care was taken not to make mistakes, perhaps especially when, as in the present text, radii were taken equal to unity and trigonometric functions were simply represented by straight-line segments. Spherical triangles may presumably have offered the advantage of drawing attention more automatically to the center of the arcs whose trigonometric functions came into play, as well as to the arcs themselves.

As central angles have the same measure as their arcs, or, rather, as they are measured by their arcs, there is of course nothing odd in drawing attention to this point. Likewise, as sines are ratios, in comparing different sines it is necessary to refer them to denominators of the same value when the denominators are missing. It is therefore reasonable and even necessary to be careful to adhere to this principle when the denominators of the fractions representing these ratios are kept in the background and are not explicitly referred to, as is the case in the present tract. The point here is that such precautions seem to be taken when applying trigonometry to plane triangles and not in applying it to spherical triangles. And this suggests that plane trigonometry was a less familiar subject as compared to spherical trigonometry.

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It would seem worthwhile to summarize the content of our text by presenting summaries of Abû Naṣr's proofs, as transmitted by Al-Bayrûnî in this letter, in modern notations. We have the following five proofs for the five special cases dealt with in the text:

1 — In the first case (figure 1) we want to prove that in the spherical triangle ABG, where the angle A is a right angle, the relation $\frac{\sin AB}{\sin BG} = \frac{\sin G}{\sin A}$ holds. It is first shown that the triangles EDB and ZLH are similar. The angles GZH, GZT, HLZ, BDZ, and BEZ are right angles by construction. From the similarity of the triangles EDB and ZLH it follows that $\frac{BD}{BE} = \frac{HL}{HZ} \dots \dots \dots (1)$

$BD = \sin AB$, $BE = \sin BG$, $HL = \sin G$, and $HZ = \sin A$. Replacing therefore the terms occurring in relation (1) by their equals, we have: $\frac{\sin AB}{\sin BG} = \frac{\sin G}{\sin A}$.

Q. E. D.

2 — In the triangles ABG of figures 2 and 3 we wish to prove the same relation. This time the triangle is not a right triangle. Our proof is therefore to apply to the cases of acute-angled and obtuse-angled triangles both.

We draw the arcs BD perpendicular to the arcs AG and AD, respectively. In the spherical right triangles ABD and BDG we have $\frac{\sin AB}{\sin BD} = \frac{\sin D}{\sin A}$ and $\frac{\sin BD}{\sin BG} = \frac{\sin G}{\sin D}$ by virtue of our theorem of the first special case just demonstrated. Multiplying these two proportionalities term by term, we obtain $\frac{\sin AB}{\sin BG} = \frac{\sin G}{\sin A}$.

Q. E. D.

3 — In the plane triangle ABG of figure 4 it is required to prove that $\frac{AB}{AG} = \frac{\sin G}{\sin B}$.

In case the angle G is a right angle, then, $AB = \sin G$ and $AG = \sin B$. Dividing both sides of these equalities term by term respectively and setting them equal, we obtain $\frac{AB}{AG} = \frac{\sin G}{\sin B}$.

Q. E. D.

4 — In case the triangle ABG is an acute-angled plane triangle (figure 4), we have in the right triangles AGH and AHB $\frac{AG}{AH} = \frac{\sin H}{\sin G}$

and $\frac{AH}{AB} = \frac{\sin B}{\sin H}$. Multiplying these term by term, we obtain

$$\frac{AG}{AB} = \frac{\sin B}{\sin G}.$$

Q. E. D.

5 — In the triangle AGD it is to be proved that $\frac{AG}{AD} = \frac{\sin D}{\sin G}$.

The triangle AGD is an obtuse-angled plane triangle (figure 5). DG is produced beyond G and the perpendicular AB is dropped on it. In the right triangles AGB and ABD we have

$$\frac{AG}{AB} = \frac{\sin B}{\sin G} \quad \text{and} \quad \frac{AB}{AD} = \frac{\sin D}{\sin B}.$$

Multiplying these, we obtain $\frac{AG}{AD} = \frac{\sin D}{\sin G}$.

Q. E. D.

رسالة ابي الريحان البيروني

في الشكل المغني وطريق استخراج دعاوى هذا القانون

لابي نصر منصور بن علي بن عراق*

بسم الله الرحمن الرحيم . صلى الله على سيدنا محمد وآله وصحبه وسلم .
استخراج الشيخ الفاضل ابي نصر منصور بن علي مولى امير المؤمنين
ايده الله .

مثلث ABG على الكرة من الدوائر العظام وزاوية A قائمة . {شكل 1} .
اقول ان نسبة جيب AB الى جيب BG كنسبة جيب زاوية G الى جيب
زاوية A .

برهانه انا تم ربعي الدائرة وهما GBH ، GAT ونرسم قوس HT
التي هي بمقدار زاوية G [ونصل نقط H ، T ، A ، G بمركز الكرة
وهو Z بخطوط HZ ، TZ ، AZ ، GZ المستقيمة ونخرج جيوب
HL ، BD ، BE ونصل ED .

فن اجل ان AZ الفصل المشترك لسطحي دايرتي GAT ، AB و BD
عمود على AZ فانه عمود على سطح دائرة GAT ولذلك ايضا HL
عمود على هذا السطح اعني سطح دائرة GAT فـ HL ، BD متوازيان
ولان HG ربع دائرة فان زاوية HZG قائمة وكذلك زاوية BEZ
قائمة و HZ ، BE في سطح واحد لانها في سطح دائرة GBH فـ HZ ، BE
متوازيان فسطح مثلث DBE يوازي سطح مثلث LHZ ويفصلهما

134b
135a

* Bu kısım bu yayında yapılmış bir ilâvedir. Yazma metninde başlık yoktur.
(No such heading occurs in the manuscript text. It has been added here merely
for convenience).

سطح دائرة GAT لان نقط D، E، L، Z في هذه الدائرة فثلاثا ¹HLZ،
 BED متشابهان فنسبة BD الذي هو جيب AB الى BE الذي هو جيب
 BG كنسبة HL الذي هو جيب زاوية G الى HZ الذي هو جيب زاوية
 A. { شكل 1 } .

ولتكن ايضا زاوية [A غير ال (؟)] قائمة ². فنقول انه تكون نسبة
 جيب AB الى جيب BG كنسبة {جيب} زاوية G الى {جيب} زاوية A.
 برهان ذلك انا نخرج عمود BD من دائرة عظيمة على AG فلان
 نسبة جيب AB الى جيب BD كنسبة جيب زاوية D الى جيب زاوية
 A ونسبة جيب BD الى جيب BG كنسبة {جيب} زاوية G الى
 {جيب} زاوية D فبالمساواة في النسبة المضطربة تكون نسبة جيب AB
 الى جيب BG كنسبة جيب زاوية G الى جيب زاوية A وذلك ما اردنا
 بيانه . { شكل 2 ، 3 } .

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مثلث ABG مفروض . { شكل 4 } . اقول ان نسبة AB الى AG
 كنسبة جيب زاوية G الى جيب زاوية B.

برهانه : ينبغي اولا ان يعلم من قولنا جيب زاوية كذا انا نريد
 في المثلث من الخطوط المستقيمة جيب القوس التي تكون على تلك الزاوية
 اذا كانت الزاوية على مركز دائرة ، وبعد [ذلك ؟] نجعل نقطة B
 مركزا وندير عليه ببعد BA قوس AD ونخرج [خط BG من]
 نقطة G على استقامة ، وليلقها على نقطة D.

فان كانت زاوية G قائمة فان AG جيب AD و AD بمقدار زاوية
 B التي على المركز ولان AB نصف قطر هذه الدائرة فانه جيب
 زاوية G التي اذا كانت على المركز كان الذي يوترها من الدائرة ربعها .
 فقد صح الدعوى اذا كانت G قائمة . { شكل 4 } .

1.HKZ

² زاوية A ليست بقائمة (؟) .

ثم نجعلها غير قائمة ونخرج من نقطة A على خط BD عمود AH. فنسبة AG الى AH كنسبة جيب زاوية H القائمة الى جيب زاوية G وذلك ان AH اذا كان ذلك كذلك يكون جيب زاوية G في الدائرة التي نصف قطرها AG ، ونسبة AH الى AB كنسبة جيب زاوية B الى جيب زاوية H القائمة لان AH يكون جيب زاوية B في الدائرة التي نصف قطرها AB. فبالمساواة في النسبة المضطربة نسبة AG الى AB كنسبة جيب زاوية B الى جيب زاوية G. {شكل 4} .

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عبارة اخرى : مثلث AGD مفروض . {شكل 5} . اقول ان نسبة خط AG الى خط AD من مثلث AGD كنسبة جيب زاوية D الى جيب زاوية G.

برهانه انا نخرج DG ، AB حتى يلتقيا على نقطة B على زاوية قائمة . فبين ان نسبة AG الى AB كنسبة جيب زاوية B الى جيب زاوية G وكذلك¹ نسبة AB الى AD كنسبة جيب زاوية D الى جيب زاوية B . فبالمساواة في النسبة المضطربة نسبة AG الى AD كنسبة جيب زاوية D الى جيب زاوية G. وذلك ما اردنا بيانه .
نسخة كتاب ابي الريحان الى ابي سعيد ، رحمها الله تعالى .

¹ ولذلك .

ARAPÇA METNİN TÜRKÇE ÇEVİRİSİ

Esirgeyen ve yarlıgayan Tanrı'nın adıyla. Efendimiz Muhammed'e, ailesi fertlerine, ve yakınlarına Tanrı dirlik ve esenlik versin.

Müminlerin Emîri Başbuğumuzun — Tanrı onu korusun ve güçlü kılsın — yakın akrabası erdemli şeyh Ebû Nasr Mansur ibn Ali'nin tanıtılma yolları.

ABG üçgeni küre üzerinde bulunup kenarları büyük daire yayları ve A açısı diktir (Şekil 1). AB'nin sinüsünün BG'nin sinüsüne oranının G açısının sinüsünün A açısının sinüsüne oranına eşit olduğunu savlıyorum.

Tanıtlanmasına gelince: GB ve GA yaylarını çeyrek daireye tamamlayalım. Meydana gelen çeyrek daire yayları GBH ve GAT'dir. G açısının ölçüsünü veren HT yayını çizelim ve H, T, A, ve G noktalarını HZ, TZ, AZ, ve GZ doğru parçaları aracılığıyla küre merkezi olan Z noktasıyla birleştirelim. Yine, sinüs olan HL, BD, ve BE doğru parçalarını çizelim ve D ile E'yi ED doğru parçasını çizerek birleştirelim.

İmdi, AZ doğrusu GAT ve AB daireleri düzlemlerinin ara kesiti olduğundan ve BD doğru parçasının AZ'ye dik olduğundan, BD zorunlulukla GAT dairesi düzlemine diktir ve aynı sebeple HL de bu düzleme, yani GAT dairesi düzlemine diktir. Böyle olunca da, HL ile BD birbirlerine paraleldirler. Öte yandan, HG bir çeyrek daire olduğundan, HZG açısı bir dik açıdır. Aynı suretle, BEZ açısı da | diktir. Ayrıca, HZ ile BE aynı düzlemedir. Çünkü her ikisi de GBH dairesi düzlemi içinde bulunmaktadır. Böylece, HZ ile BE paraleldirler ve DBE üçgeninin düzlemi LHZ üçgeninin düzlemine paraleldir. Öte yandan, bu üçgenler GAT dairesi düzlemi tarafından kesilmektedirler. Çünkü D, E, L, ve Z noktaları bu daire düzlemi içindedirler. Binaenaleyh [bu üçgenlerin ED ve LZ kenarları da paraleldir ve] HLZ ve DBE üçgenleri benzer üçgenlerdir. Demek ki, AB'nin sinüsü olan DB'nin, BG'nin sinüsü olan BE'ye oranı G

açısının sinüsü olan HL 'nin, A açısının sinüsü olan HZ 'ye oranına eşittir (Şekil 1).

Şimdi de A açısının da dik olmaması halini ele alalım (Şekil 2 ve 3). AB 'nin sinüsünün BG 'nin sinüsüne oranının, G açısının sinüsünün A açısının sinüsüne oranına eşit olduğunu beyan ediyoruz.

Bunun ispatını şu yoldan yaparız: BD büyük daire yayını AG 'ye dik olarak çizeriz. İmdi, AB 'nin sinüsünün BD 'nin sinüsüne oranı D açısının sinüsünün A açısının sinüsüne oranına ve BD 'nin sinüsünün BG 'nin sinüsüne oranı G açısının sinüsünün D açısının sinüsüne oranına eşit olduğundan, bu iki orantının taraf tarafa çarpımı AB 'nin sinüsünün BG 'nin sinüsüne oranının G açısının sinüsünün A açısının sinüsü oranına eşitliği bağıntısını verir. Böylece de teoremimiz ispatlanmış olur (Şekil 2 ve 3).

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Bu kez ABG üçgeni verilmiş olsun (Şekil 4). AB 'nin AG 'ye oranının, G açısının sinüsünün B açısının sinüsüne oranına eşit olduğunu beyan ediyorum.

Tanıtı: İlk, kenarları doğru parçalarından oluşma bir üçgene ilişkin olarak "falanca açının sinüsü" sözümüzle, bir daire merkezinde bulunduğu bu açının göreceği yayın sinüsünü ifade etmek istediğimiz bilinmelidir. Bu belirtmeden sonra, B noktası merkez olmak üzere bu nokta etrafında BA yarıçapıyla AD yayını çizer, BG doğru parçasını G noktası ötesinde uzatırız. BG 'nin bu uzantısı AD yayını D noktasında kessin.

İlkin G açısının dik olması halini ele alalım. AG doğru parçasının AD 'nin sinüsü olduğunu ve AD 'nin merkezşel B açısının ölçüsünü verdiğini biliyoruz. Ayrıca, AB doğru parçası bu daire yarıçapı olduğundan, bu doğru parçası, merkezde bulunduğu takdirde bir çeyrek daire yayını görmek durumunda olan G açısının sinüsüdür. Böylece de, G açısının dik olması şıkkı için teoremimiz ispatlanmış olur (Şekil 4).

Şimdi de G açısının dik olmadığını farz edelim. A noktasından BD doğrusuna AH dikmesini indirelim. AG 'nin AH 'ye oranı H dik açısının sinüsünün G açısının sinüsüne oranına eşittir. Bu da şundan ötürüdür ki, eldeki durumda AH doğru

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parçası, yarıçapı AG olan dairede G açısının sinüsüdür. Yine, AH'nin AB'ye oranı, B açısının sinüsünün | H dik açısının sinüsüne oranına eşittir. Çünkü AH, yarıçapı AB olan dairede B açısının sinüsüdür. Eldeki bu iki orantıyı taraf tarafa çarparsak, AG'nin AB'ye oranının, B açısının sinüsünün G açısının sinüsüne oranına eşitliği bağıntısını elde ederiz (Şekil 4).

Bir başka açıklama (veya, tümleyici bağlantı) da şudur: AGD üçgeni verilmiş olsun (Şekil 5). AGD üçgeninde, AG doğru parçasının AD doğru parçasına oranının, D açısının sinüsünün G açısının sinüsüne oranına eşit olduğunun saptanması söz konusudur.

Bu bağlantıyı ispatlamak için DG doğru parçasını A'dan indirilen AB dikmesiyle B noktasında kesişinceye kadar uzatalım. GA'nın AB'ye oranının, B açısının sinüsünün G açısının sinüsüne oranına eşitliği saptanmış bulunuyor. Aynı suretle, AB'nin AD'ye oranının, D açısının sinüsünün B açısının sinüsüne oranına eşit olduğunu da biliyoruz. Bu orantıların taraf tarafa çarpımı ise AG'nin AD'ye oranının, D açısının sinüsünün G açısının sinüsüne oranına eşitliği sonucunu verir. Böylece de aradığımız ispat elde edilmiş olur.

Bu yazı Ebû'r-Reyhan'ın Ebû Said'e mektubunun bir kopyasıdır. Yüce Tanrı her ikisine de rahmet etsin.

TRANSLATION OF THE ARABIC TEXT

Derivations found by the virtuous *shaykh* Abû Naṣr Maṣṣûr ibn 'Alî, the close relative of the Amîr of the Believers, may God support and strengthen him.

Let the triangle ABG be on the surface of a sphere, its sides being arcs of great circles, and let the angle A be a right angle. (Figure 1). I assert that the ratio of the sine of AB to the sine of BG is equal to the ratio of the sine of the angle G to the sine of the angle A.

Its proof is as follows: Let us complete GB and GA to quarters of circles. These are GBH and GAT. We draw the arc HT, which is of the magnitude of the angle G, and we join the points H, T, A, and G to Z, the center of the sphere, through the straight line segments HZ, TZ, AZ, and GZ. We then draw the sines HL, BD, and BE, and we likewise draw ED by joining E to D.

Now, because AZ is the line of intersection of the planes of the circles GAT and AB and BD is perpendicular to AZ, BD is necessarily perpendicular to the plane of the circle GAT, and for the same reason HL is also perpendicular to the same plane, i. e., to the plane of the circle GAT. HL and BD are therefore parallel, and because HG is a quarter of a circle, HZG is a right angle. Likewise, the angle BEZ is a right angle; and HZ and BE are in the same plane, for they are in the plane of the circle GBH. Therefore, HZ and BE are parallel, and the plane of the triangle DBE is parallel to the plane of the triangle LHZ. And these triangles are intersected by the plane of the circle GAT. For the points D, E, L, and Z are on the plane of this circle. Therefore, [the line segments ED and ZL are parallel, and] the triangles HLZ and BED are similar. Consequently, the ratio of DB, which is the sine of AB, to BE, which is the sine of BG, is equal to the ratio of HL, which is the sine of the angle G, to HZ, which is the sine of the angle A. (Figure 1).

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Let now the angle A too not be a right angle. (Figures 2, 3). We state that the ratio of the sine of AB to the sine of BG is equal to the ratio of the sine of the angle G to the sine of the angle A.

Its proof is as follows: We draw the arc of great circle BD perpendicular to AG. Now, as the ratio of the sine of AB to the sine of BD is equal to the ratio of the sine of the angle D to the sine of the angle A and the ratio of the sine of BD to the sine of BG is equal to the ratio of the sine of the angle G to the sine of the angle D, their products being likewise equal, one obtains the result that the ratio of the sine of AB to the sine of BG is equal to the ratio of the sine of the angle G to the sine of the angle A. (Figures 2 and 3).

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Q. E. D.

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Let next the triangle ABG be given. (Figure 4). I assert that the ratio of AB to AG is equal to the ratio of the sine of the angle G to the sine of the angle B.

As to its proof, it is necessary first of all to know that by the expression "the sine of such and such an angle" in a triangle whose sides are made up of straight-line segments we mean to refer to the sine of the arc which such an angle subtends when this angle is located at the center of a circle. We describe then with the point B as center and with AB as radius the arc AD and produce the line BG beyond the point G. Let this extension of BG meet the arc AD at the point D.

Now, in case the angle G is a right angle, AG is the sine of AD which is of the same magnitude as the angle B taken at the center of this arc. And, as AB is the radius of this circle, it is the sine of the angle G, which, if located at the center, would subtend an arc of a quarter of a circle. Thus, the theorem is verified for the case wherein the angle G is a right angle. (Figure 4).

Next we suppose G not to be a right angle. From the point A we drop the perpendicular AH upon the line BD. The ratio of AG to AH is equal to the ratio of the sine of the right angle H to the sine of the angle G, and this is because, under these

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136a circumstances, AH is the sine of the angle G in the circle whose radius is AG. Moreover, the ratio of AH to AB is equal to the ratio of the sine of the angle B to the sine of the angle H which is a right angle. For AH is the sine of the angle B in the circle the radius of which is AB. Consequently, by multiplying and setting equal both sides of these two equalities one obtains the relation that the ratio of AG to AB is equal to the ratio of the sine of the angle B to the sine of the angle G. (Figure 4).

Another exposition: Let the triangle AGD be given. (Figure 5). It is to be proved that the ratio of the side AG of the triangle AGD to its side AD is equal to the ratio of the sine of the angle D to the sine of the angle G.

Its proof: We produce DG and drop from A a perpendicular to it. Let them meet at B. It is clear that the ratio of AG to AB is equal to the ratio of the sine of the angle B to the sine of the angle G. Likewise, the ratio of AB to AD is equal to the ratio of the sine of the angle D to the sine of the angle B. By multiplying and setting equal both sides of these two equalities, we have: the ratio of AG to AD is equal to the ratio of the sine of the angle D to the sine of the angle G.

Q. E. D.

This is the copy of Abū ar-Rayḥān's letter to Abū Sa'īd, may God, He is exalted, have mercy on them both.