HÜSEYİN TEVFİK PASHA – THE INVENTOR OF 'LINEAR ALGEBRA'

Gert Schubring[*]

Linear algebra constitutes today one of the most important basic theories of modern mathematics. During the period of the curricular reform movement, also called "modern mathematics", linear algebra even replaced proper geometry teaching within the school curriculum. It is revealing to study the origins of the term of "linear algebra", and of its theory.

While comparing mathematical cultures, it might be useful to comment on developments at the "periphery",[1] where their innovations often go beyond the state of the art attained in the "metropoles," even though these innovations may be noticed indirectly at best. While I was participating in a conference on the history of science in Istanbul in 1991, a 1988 reedition of a textbook *Linear Algebra* which had originally been published in 1882 by Hüseyin Tevfik Pasha (in English!) caught my eye. The author had published a second edition in 1892. Both versions were reprinted in the 1988 edition. The book and its author, entirely unknown until the reedition even in Turkey, were rediscovered when a copy of the 1892 version appeared in the catalogue 202 of 1985 of the German antiquarian bookseller Sändig, a well known specialist for old mathematical books. It is hence probable that the original owner had been a German mathematician who had bought it in Istanbul. It had been highly difficult for Kâzım Çeçen, the editor of the reedition, to find copies of the 1892 publication in Turkey - and of the first one, there was only one copy identifiable (Çeçen 1988, p.16).

When I tried to find out later at which point in time the term of *linear algebra* – which was so decisive for the development of mathematics in the 20th century – had been first used, I was informed by Gregory Moore that van der Waerden was the first to use this term.[2] For the nineteenth century, the use of "linear" seemed only to be known in the connection with "associative" - disseminated by the well known book "Linear Associative Algebra" published in 1870 by Benjamin Peirce.[3] Actually, it proved that A.N. Whitehead had mentioned the term "linear algebra" in 1898 when he announced it as the

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* University of Bielefeld.

1 For the concept of relation between "metropole" and "periphery", see Lewis Pyenson 1989.

2 See also Moore (1995, p.294), where he mentions an earlier use of the term by H. Weyl in 1918.

3 Benjamin Peirce (1809-1880), the father of Charles Saunders Peirce, was a mathematician and astronomer who decisively contributed to promoting mathematics, and in particular research on algebra, in the USA. From 1833 on, he was a mathematics professor at Harvard.

subject of the intended second volume of his famous *Treatise on Universal Algebra* (Whitehead 1898, p.v)

Hüseyin Tevfik Pasha (1832-1901), educated at the'Mühendishane' (Military School of Engineering, Istanbul), was active there and in private endeavours of teaching mathematics and the sciences. As military *attacheé* in France from 1868 to 1870, he improved his knowledge in mathematics. Another stay abroad, from 1872 to 1880 in the United States, was decisive for introducing him to mathematical research. It seems that he was in contact with Peter Guthrie Tait (1831-1901),[4] who painstakingly watched over maintaining the orthodoxy of the Hamiltonian school, but he achieved results independently. After his return, Tevfik Pasha was highly active in Ottoman society to promote learning and the sciences. He became Rector Magnificus of the Military Academy at Istanbul, the *Mühendishâne*, and served as minister in the government (Çeçen 1988, p.15).

The "contemporary assessment" which Cahit Arf gives as an introduction to the 1988 reedition is somewhat misleading, since it gives the impression as if Tefvik Pasha's primary source and motivation had been Hamilton's theory of quaternions, and that he had aimed to reduce Hamilton's four-dimensional approach to a three-dimensional one - so that the non-associativity of Tefvik's concept would have been a consequence of that transposition of the quaternions to three dimensional space.

Actually, there is a certain reverence for Hamilton as founder of the quaternion theory, but Tevfik Pasha claims already there to have developed a simpler theory. And the true source of inspiration is revealed only in the second edition: it is Argand's concept of a vector calculus.

While the author does not speak, in his first edition, about his motivations, he does so, extensively, in the Preface to the second edition:

"Linear Algebra, as treated in this pamphlet, grew out of an effort to extend Argand's system concerning ordinary complex or imaginary quantities to space of three dimensions".

Altough Hamilton's quaternions grew out of a similar effort, the author says, the two systems have little in common: Argand's system was not a special case of Hamilton's and Argand's system was incomplete, being restricted to two dimensions. The author shows that he is acquainted with Cauchy's use of Argand's ideas, and with Bellavitis' equipollences. He claims to have established "a new Algebra", completing Argand's in particular with regard to multiplication (Tevfik Pasha 1892, p.5). One might wonder whether this

[4] Some examples refer to one of Tait's textbooks, e.g. Tevfik Pasha 1882, p.60 f.). In Chapter I, he pointed for more complicated problems to the textbook Introduction to Quaternions by Kelland and Tait.

referring to Argand applied in fact also for the first edition or whether it was introduced only subsequently. As the author makes clear, at least F.J. Servois's papers of 1813 and 1814 were discussed by the Tait school,[5] and these papers not only referred to Argand's work, but already tried to generalize it. On the other hand, Tevfik Pasha refers as source for Argand's work to its English edition by A.S. Hardy. Actually, Argand's paper of 1806 had been rediscovered by Jules Houël and published with a commentary in 1872.[6] And this French edition had been translated and again commented by Hardy in 1881. It is possible that Tevfik Pasha had already obtained the French version of 1874, and that the English version of 1881 had only provided him with some additional ideas. One can therefore legitimately assume that Tevfik Pasha's claim to have been motivated by Argand is basically correct. On the other hand, Argand's main merit is to have established the geometrical representation of imaginary quantities and the proof of the fundamental theorem of algebra, and not so much as a pioneer of vector calculus. There are other proponents after him who did so more explicitly, like C.V. Mourey, J. Warren, G. Bellavitis, - and Hermann Graßmann, evidently.

As Tevfik Pasha explains, he understands "linear algebra" as juxtaposed to "numerical algebra" (Tevfik Pasha 1882, p.15): whereas numerical or ordinary algebra deals with numbers or numerical quantities, linear algebra deals with lines. Tevfik Pasha's idea, hence, constitutes another approach to establish algebraic operations on geometric quantities! His approach is therefore directly comparable to that of Hermann Graßmann; it is quite evident, however, that Tefvik Pasha had neither read Graßmann's *Ausdehnungslehre*, nor heard of it.

In fact, in both editions, the author proceeds by establishing a basic calculus with vectors, namely to establish addition of oriented lines - i.e. vectors - and than subtraction. The crucial point for all such approaches was how to define multiplication. The result of multiplying two lines gives another line not coplanar to them, so that the three lines are elements of a three-dimensional space. This vector product proves to be an exterior - or outer - product. As such, it is non-commutative, in general (Tevfik Pasha 1882, p.11 ff.). There is also an inner product, resulting in a scalar, but it is mentioned without name just in an example (Tevfik Pasha 1892, p.175).

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5 Tevfik Pasha mentions the geometry textbook by De Volson Wood as reflecting about the rlation between Argand and Servois (Tevfik Pasha 1892, p.6).

6 Since the findings published in my own contribution to the Wessel Conference of 1998 are still not largely disseminated, it should be repeated here that the biographical data given by Houël in 1872 and always repeated since then, are entirely false; neither was his name Jean Robert Argand, nor are his dates of death and birth 1768 and 1822 respectively. His first name(s) are as unknown as his dates of life. The only legitimate statement about his biography is: Argand, fl. 1806, 1813, 1814! See Schubring 2001.

The second edition of 1892 is more theoretic and more structured according to the exigencies of axiomatisation. Tevfik Pasha looks for the reverse of multiplication and he is clear that for an ordinary algebra the "inverse operation" of multiplication would be needed. Due to the non-communtativity of multiplication, there is, however, no unique inverse operation - no "division" (Tevfik Pasha 1892, p 29). Contrary to Graßmann, the author does not invest more energy to establishing operations to fill the lacuna.

It is also in the second edition that the author asks "whether the Associative Law holds true", for "three lines in space". The answer is: in general, it does not, but it is formulated by the author in much more positive terms. He says that the multiplication is associative if certain conditions are fulfilled - and the conditions are such that in the case of what he calls "Argand's Algebra"[7] the multiplication is associative, i.e. if the three lines are coplanar and in the same plane as the principal axis (ibid., p.27 f.). The second edition is enlarged, in particular by additional applications.

In both editions, he introduces and discusses a third type of multiplication, besides numerical and linear multiplication. This third type is called by the author "complex multiplication" and consists in operating with "complex quantities", i.e. "symbolical expressions compounded of numerical quantities and lines having their directions as well as their lengths" (ibid., p.157). A typical expression is

$A+\alpha = B + \beta$, implying $A=B$ and $\alpha = \beta$.

For the multiplication of these quantities, he introduces the new sign $\cap$: $p \cap r$ (known to us from Graßmann).

And for this type, he clearly states "complex multiplication is not generally commutative" (ibid., p.159). Furthermore, he states that this operation is distributive but is in general not associative - again formulated as associative under a number of conditions (ibid., p.168f.).

Tevfik Pasha applies his Linear Algebra mainly to plane geometry: to deal with the conics - circle, ellipse, hyperbola, parabola -, but also to three-dimensional objects like the sphere and the cone, to develop their fundamental qualities and classical propositions (like Pascal's theorem) by a coordinate-free approach.

In the 1892 edition, the author introduced a final section where he compared the procedures for solving basic questions either by his linear algebra, or by his complex operating or by Hamilton's quaternions. The intended result is that both his own systems proceed more simply than Hamilton's system.

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[7] or "De Morgan's Double Algebra".

One can understand Tevfik Pasha's notion “linear algebra” as originating from an approach aiming at generalizing the notion of multiplication to lines in the two- and the three-dimensional case, thus establishing a version of vectorial calculus. His focus on Argand as his source of motivation was conditioned by the lens of reception as practiced by Tait’s school of quaternionists.

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